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Localised vibration modes in free anisotropic wedges

A. L. Shuvalov and V. V. Krylov

Department of Civil and Structural Engineering, The Nottingham Trent University, Burton Street, Nottingham NG1 4BU, United Kingdom

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Propagation of flexural localised vibration modes along edges of anisotropic wedges is considered in the framework of the geometrical-acoustics approach. Its application allows for straightforward evaluation of the wedge-mode velocities in the general case of arbitrary elastic anisotropy. The velocities depend on the wedge apex angle and on the mode number in the same way as in the isotropic case, but there appears to be additional dependence on elastic coefficients. The velocities in tetragonal wedges (with the midplane orthogonal to the four-fold axis) and in ‘‘weakly’’ monoclinic wedges are explicitly calculated and analyzed. Bounds of the wedge-wave velocity variation in tetragonal materials are established. © 2000 Acoustical Society of America.

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INTRODUCTION

The existence of antisymmetric localised waves propagating along edges of elastic solid wedges in contact with vacuum was predicted in 1972 by Lagasse and Maradudin et al. By using numerical calculations, it was shown in Refs. 1 and 2 that such waves, now often called wedge acoustic waves, have low propagation velocities (generally much lower than that of Rayleigh waves), and that their elastic energy is concentrated in the area of about one wavelength near the wedge tips. Since 1972, wedge acoustic waves have been investigated in a number of papers with regard to their possible applications to signal processing, nondestructive testing of special engineering constructions, structural dynamics, etc. (see, e.g., Refs. 3–12).

Because of the complexity of the problem even for wedges made of elastically isotropic materials, no exact analytical theory of the localised wedge modes is available which could describe their propagation for an arbitrary wedge apex angle. However, for slender wedges, the approximate analytical theory has been developed on the basis of the geometrical acoustics approach, which interprets the problem in terms of the quasiplane flexural waves propagating in a plate of variable thickness. In Ref. 5 the case of elastic isotropy in the wedge midplane has been envisaged (for brevity, we will be referring to such transversal isotropy as simply the isotropy). Reference 6 has been concerned with a slender wedge of weakly anisotropic material, whose departure from the isotropic state can be described by a single (and small) parameter. In this framework it was demonstrated that weak anisotropy does not affect the dependence of wedge-wave velocities on the wedge apex angle and the mode number.

In the present paper, we consider antisymmetric localised waves in a generally anisotropic slender wedge. The relation for wedge-wave velocities is obtained which reveals the same dependence on as in the isotropic case, but acquires additional dependence on elastic coefficients. As an example, the velocities are explicitly calculated and analyzed for tetragonal wedges, with the midplane orthogonal to the four-fold axis, and also for ‘‘weakly’’ monoclinic wedges.

I. CONSIDERATIONS FOR AN ARBITRARY ANISOTROPY

According to Refs. 5 and 6, the velocities of antisymmetric localised modes in an elastic wedge with a small apex angle are defined by the equation

$$\int_{c} [k_n^2(x, \varphi) - \beta^2]^{1/2} dx = 2\pi n \quad (n = 1, 2, \ldots),$$

(1)

where is the mode order,

$$\beta = k_n(x, \varphi) \cos \varphi,$$

(2)

is the polar angle varying in the midplane of the wedge, is the wave number of flexural mode in a plate of the small
variable thickness \( h = \theta x \) (Fig. 1). The integration path \( C \) follows the ray trajectory, which starts from the edge \( x = 0 \) where \( \varphi = \pi/2 \), passes the turning point \( x_t \) corresponding to \( \varphi = 0 \), and ends up at the edge \( x = 0 \) having \( \varphi = -\pi/2 \). In the case of isotropic material,

\[
k^2_0(x) = \frac{2\sqrt{3} \omega}{\theta x c_{0p}},
\]

where \( c_{0p} \) is the speed of the longitudinal dilatational mode in an isotropic thin plate. This leads to the result of Ref. 5 for the antisymmetric-mode velocity in the isotropic wedge,

\[
e^{(iso)} = c_{0p} \frac{\theta n}{\sqrt{3}}.
\]

It can be shown\(^{13}\) that in the case of generally anisotropic material with the density \( \rho \) and the tensor of elastic moduli \( c_{ijkl} \), the squared value \( k^2_0(x, \varphi) \) becomes

\[
k^2_0(x, \varphi) = \frac{2\sqrt{3} \omega}{\theta x \sqrt{f(\varphi)}},
\]

where

\[
f(\varphi) = \frac{1}{\rho} \mathbf{m} \cdot [(nn)(mm) - (mn)(nm)^{-1} + (nm)] \mathbf{m},
\]

\( \mathbf{m} = \mathbf{m}(\varphi) \) is the unit vector turning about the angle \( \varphi \) in the midplane, \( \mathbf{n} \) is the unit normal to the midplane, and \( \ldots \) are the matrices written by means of notation

\[
(a b) = a_c c_{ijkl} b_k \text{ for any vectors } a, b.
\]

In the generic case, \( f(\varphi) \) (6) is a homogeneous polynomial of the 4th degree in \( \sin \varphi, \cos \varphi \) with the coefficients depending on elastic moduli. Because the matrix, enclosed in brackets in (6), is positive-definite for an arbitrary tensor \( c_{ijkl} \) and any vectors \( \mathbf{m}, \mathbf{n} \), the quadratic form \( f(\varphi) \) is positive and so the flexural-mode wave number (5) is well-defined for an arbitrary anisotropy.

In the presence of anisotropy, conjunction of (2) and (5) cannot provide explicit \( \varphi(x) \), but it readily yields \( x(\varphi) \).

\[
x(\varphi) = \frac{2\sqrt{3} \omega \cos^2 \varphi}{\theta^2 \beta^2 \sqrt{f(\varphi)}}.
\]

It is therefore convenient to pass to the integration over \( \varphi \), writing Eq. (1) with the aid of (2) as

\[
\beta \int_{\pi/2}^{-\pi/2} x'(\varphi) \tan \varphi d\varphi = 2\pi n.
\]

Inserting (8) into (9) results in the following relation for the wedge-wave velocities \( c \):

\[
c = \frac{\theta n \pi}{\sqrt{3} J},
\]

where

\[
J = \int_{\pi/2}^{-\pi/2} \left[ f^{-1/2}(\varphi) \cos^2 \varphi \right]' \tan \varphi d\varphi,
\]

and \( f(\varphi) \) is defined by (6). The obtained relation describes the velocities \( c \) in an arbitrary anisotropic wedge. It may also be cast into the form

\[
c = c^{(iso)} \frac{\pi}{c_{0p} J},
\]

where \( c^{(iso)} \) is given by (4). The dimensionless coefficient \( (\pi J/c_{0p} J) \) shows the anisotropy-induced change of the wedge-wave velocities with respect to the reference (transversely) isotropic medium, characterized by the dilatational plate-mode speed \( \sqrt{f(\varphi_{ref})} = c_{0p} \), where a polar angle \( \varphi_{ref} \) may be chosen optionally. Once picked, \( \varphi_{ref} \) sets \( c_{0p} \) and, thereby, specifies a particular reference isotropic medium.

Comparing (4) and (10), (12) confirms that the impact of unrestricted elastic anisotropy does not affect the dependence of velocities \( c \) on the apex angle \( \theta \) and mode number \( n \). It is also noteworthy that \( c \) does not depend on the orientation of the wedge edge in the given midplane due to the period \( \pi \) of \( f(\varphi) \) (6), which merely reflects the inversion symmetry of elastic-wave properties in an arbitrary anisotropic medium.

**II. TETRAGONAL WEDGE**

As a simple example, consider a tetragonal wedge, whose midplane is orthogonal to the four-fold axis. It is convenient to choose the edge orientation so that \( \varphi = 0 \) corresponds to the direction of the two-fold axis in the midplane. In this case, (6) simplifies to

\[
f(\varphi) = f_0 \left[ 1 - \frac{C}{2 f_0 \sin^2 2 \varphi} \right],
\]

where

\[
f_0 = \frac{1}{\rho} \left( c_{11} - \frac{c_{13}^2}{c_{33}} \right), \quad C = \frac{1}{\rho} (c_{11} - c_{12} - 2 c_{66}),
\]

\( C \) being the parameter of anisotropy which measures the deviation of a given tetragonal material from the transversely isotropic state.

Positive-definiteness of elastic strain energy leads to the condition\(^{14}\)

\[
(c_{11} + c_{12} - 2 c_{13}^2) > 0,
\]

which in turn demands that \( f_0 > 0 \) and \( C < 2 f_0 \). Thus the dimensionless parameter \( C/2 f_0 \) is always less then one, and

FIG. 1. Geometry of the problem.
may take either positive or negative values in different anisotropic materials. Strictly speaking, \( C/2 f_0 \) is not bounded from below. However, invoking (15) reveals that the inequality \( c_{66} > c_{11} \) is the necessary condition for \( C/2 f_0 \) \(< -1 \). Shear moduli exceeding extensional ones is quite a rare occasion, so the low bound \( C/2 f_0 > -1 \) is well applicable for almost all tetragonal materials.

Substituting (13) into (11) gives

\[
J = \frac{1}{\sqrt{f_0}} \int_{-\pi/2}^{\pi/2} \left\{ \frac{\cos \varphi}{[1 - (C/2 f_0) \sin^2 \varphi]^3} \right\} \, d\varphi
\]

which is obviously positive by virtue of \( C/2 f_0 < 1 \). Provided that the parameter of anisotropy \( C \) in a given material is positive,

\[
J = \frac{2}{\sqrt{f_0}} F(k, \pi/2), \quad k = \sqrt{\frac{C}{2 f_0}},
\]

where \( F(k, \pi/2) \) is the full elliptical integral of the first order,

\[
F(k, \pi/2) = \int_{0}^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}},
\]

which is defined at \( 0 \leq k^2 < 1 \). It equals \( \pi/2 \) at \( k = 0 \) and monotonically increases toward a logarithmic pole at \( k^2 \to 1 \). If \( C \) is negative, \( J \) may be written in the form

\[
J = \frac{1}{\sqrt{f_0}} \int_{0}^{\pi/2} \frac{d\varphi}{[1 - (C/2 f_0) \sin^2 \varphi]^3}
\]

Inserting (17) or (19) into (10), we may present the wedge-wave velocities \( c \) in terms of the elliptical integral \( F(k, \pi/2) \) in two alternative forms, depending on whether the parameter of anisotropy \( C \) is positive or negative, namely,

\[
c = \frac{\pi \theta n}{2 \sqrt{3} F} \sqrt{\frac{f_0}{\sqrt{2 f_0} \frac{C}{\pi}}}, \quad C \geq 0,
\]

\[
c = \frac{\pi \theta n}{2 \sqrt{3} F} \sqrt{f_0 - (C/2 f_0)} \frac{C}{\sqrt{C - 2 f_0}}, \quad C \leq 0.
\]

Let us analyze the obtained result. At \( C = 0 \), i.e., when the material is transversely isotropic, \( f_0 \) in (14) is equal to the dilatational plate-mode speed \( c_{0P} \), and Eqs. (20), (21) merge into the relation (4) obtained in Ref. 5. This limit may be naturally regarded as the reference state for evaluating the effect of tetragonal anisotropy.

Provided that anisotropy is weak, i.e., \( |C|/2 f_0 \ll 1 \), either of the equations (20), (21) yields the following approximation for \( c \):

\[
c \approx \theta n \sqrt{\frac{f_0}{3} \left[ 1 - \frac{1}{4} \left( \frac{C}{2 f_0} \right) - \frac{5}{64} \left( \frac{C}{2 f_0} \right)^2 \right]}, \quad (22)
\]

which corrects the corresponding result of Ref. 6. Equation (22) shows that the increase of positive parameter \( C \) stimulates the decrease of the wedge-wave velocities \( c \), while the increase of the absolute value of \( C \) entails increase of \( c \) (provided that \( f_0 \) is kept constant). Note that, by (22), the effect of (weak) anisotropy on the velocities \( c \) is stronger in case of \( C > 0 \).

In view of \( \pi/2 \leq F(k, \pi/2) \leq \pi/2 \) at \( 0 \leq k^2 < 1 \), the same conclusions follow from Eqs. (20), (21) which evaluate the impact of an arbitrary measure of tetragonal anisotropy. In the limit \( C/2 f_0 \to 1 \), the flexural plate-mode velocity,

\[
c_a(\varphi) = \sqrt{\frac{F(\varphi)}{12 k_a \theta x}}, \quad (23)
\]

tends to zero for the direction \( \varphi = \pi/4 + \pi l / 2 \) \( (l = 0, 1, \ldots) \), and so do the wedge-wave velocities due to the presence of the logarithmic pole in the denominator of (20). If \( C \) is negative and \( -1 < C/2 f_0 < 0 \), the argument of \( F \) in (21) varies from 0 to 1/2, which confirms that the effect of anisotropy with a given \( |C| \) is relatively stronger in case of positive \( C \).

Consider the bounds for the wedge-wave velocities. According to (13), at \( \varphi = \pi l / 2 \) and \( \varphi = \pi / 4 + \pi l / 2 \), the function \( f(\varphi) \) reaches its extremal values,

\[
f_{ext}(\varphi) = f_0; \quad f_0 - (C/2), \quad (24)
\]

being, respectively, maximum and minimum at \( C > 0 \), and vice versa at \( C < 0 \). Suppose that \( C \geq 0 \). Combining (20) with (24), and recalling that the elliptical integral \( F \) which appears in (20) is bounded from behind by \( \pi/2 \), we obtain the inequality

\[
\theta n \sqrt{\frac{f_{\min}}{3}} \leq c \leq \theta n \sqrt{\frac{f_{\max}}{3}}, \quad (25)
\]

where \( f_{\max} = f_0 \) and \( f_{\min} = f_0 - (C/2) \) at \( C \geq 0 \). If \( C \leq 0 \), then manipulating (21) leads to the same inequalities (25), in which now \( f_{\max} = f_0 - (C/2) \) and \( f_{\min} = f_0 \). It can also be shown\(^{13}\) that the extremal values of the velocity \( c_p(\varphi) \) of the dilatational mode in a thin tetragonal plate are

\[
[c_p(\varphi)]_{ext} = \sqrt{f_0}; \quad \sqrt{f_0 - (C/2)}, \quad (26)
\]

Thus the bounds (25) for wedge-wave velocities are proportional to the extrema of the velocities of the dilatational plate-mode, so that

\[
\frac{\theta n}{\sqrt{3}} [c_p(\varphi)]_{\min} \leq c \leq \frac{\theta n}{\sqrt{3}} [c_p(\varphi)]_{\max}. \quad (27)
\]

Both equalities in (27) are obviously associated with the case of transverse isotropy \( C = 0 \). In the formal limit \( f_0 \to 0 \) (hence \( C \to 0 \) due to \( C/2 f_0 < 1 \)), which implies that the flexural plate-mode velocity \( c_a(\varphi) \) tends to zero in any direction \( \varphi \), both bounds in (27), naturally, tend to zero as well. Also,
as mentioned above, $c \rightarrow 0$ at $C/2f_0 \rightarrow 1$. Note that the wedge-wave velocity bounds may be also presented via extrema of the flexural plate-mode velocity, see (23), (24).

**III. MONOCLINIC WEDGE**

Consider a monoclinic wedge with the midplane orthogonal to the symmetry plane. Bearing in mind that the wedge-wave velocities are independent of the edge orientation, we choose to measure the polar angle $\varphi$ from the direction orthogonal to the symmetry plane. Then the function $f(\varphi)$ (6) may be written in the form

$$f(\varphi) = a_1 \cos^4 \varphi + a_2 \cos^2 \varphi \sin^2 \varphi + a_3 \sin^4 \varphi$$

$$= \tilde{f}_0 \left( 1 - \frac{C}{2 \tilde{f}_0} \sin^2 \theta \cos 2 \varphi \right),$$

where, at taking crystallographic axes $X_1$, $X_2$ as, say, orthogonal ($\varphi = 0$) and parallel ($\varphi = \pi/2$) to the edge, respectively,

$$a_1 = \frac{1}{\rho} \left[ c_{44} - \frac{c_{33}^2}{c_{33} + c_{44}} \right];$$

$$a_2 = \frac{1}{\rho} \left[ c_{22} - \frac{c_{23}^2}{c_{23} + c_{44}} \right];$$

$$a_3 = \frac{2}{\rho} \left[ c_{12} + 2c_{66} - \frac{c_{45}^2}{c_{55}} \right]$$

$$- \frac{c_{14}(c_{33}c_{24} - c_{23}c_{44}) + c_{13}(c_{23}c_{44} - c_{24}c_{34})}{c_{33}c_{44} - c_{34}^2},$$

and

$$\tilde{f}_0 = \frac{1}{2} (a_1 + a_2), \quad \frac{\tilde{C}}{2 \tilde{f}_0} = \frac{a_1 + a_2 - a_3}{2(a_1 + a_2)}.$$

It is seen that the effect of monoclinic anisotropy is characterized by two dimensionless parameters ($\tilde{C}/2 \tilde{f}_0$) and $b$. The first one is similar to the parameter ($C/2f_0$) introduced above for the case of tetragonal symmetry, whereas the second one describes specifically the departure of the given monoclinic medium from the tetragonal state. Imposing no restrictions on the parameter ($\tilde{C}/2 \tilde{f}_0$), let us suppose that the parameter $b$ is small which is often valid for monoclinic and orthorhombic materials. Inserting (28) into (11) and evaluating the integrals to the first order in $b \ll 1$, we obtain the relation for the wedge-wave velocities in exactly the same form (20), (21), but with the parameters $f_0$ and ($C/2f_0$) replaced, respectively, by $\tilde{f}_0$ and ($\tilde{C}/2 \tilde{f}_0$). This shows that the impact of departure of monoclinic anisotropy from the tetragonal one reveals itself only in the second order in small parameter $b$.

**IV. CONCLUSIONS**

Conjunction of the geometrical acoustics approach with the plate theory yields the positive-definite relation for the velocities of localized vibration modes in a slender wedge of an arbitrary elastic anisotropy. The velocities retain the same dependence on the wedge apex edge and the mode number as in the isotropic case, but acquire an additional factor depending on elastic coefficients. Explicit analysis is carried out for tetragonal wedges with the midplane orthogonal to the four-fold axis, and for “weakly” monoclinic wedges. In particular it is shown that the bounds of the wedge-wave velocity value appear to be proportional to extrema of the plate-mode velocities.

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